

MATHEMATICS

ANSWER KEY

SECTION A – 20 MARKS

Question 1

In answering Multiple Choice Questions, candidates have to write either the correct option number or the explanation against it. Please note that only ONE correct answer should be written.

- (i) **(d) reflexive but not symmetric** [1]

Reflexive: $\forall a \in \mathbb{Z}$, we have,

$$a^2 - 7a \cdot a + 6a^2 = a^2 - 7a^2 + 6a^2 = 0 \Rightarrow (a, a) \in R$$

$\therefore$ Relation is reflexive

Symmetric:

For (6,1) as, $6^2 - 7 \times 6 \times 1 + 6 \times 1^2 = 36 - 42 + 6 = 0 \Rightarrow (6,1) \in R$.

But, for (1,6): $1^2 - 7 \cdot 1 \cdot 6 + 6 \cdot 6^2 = 1 - 42 + 216 \neq 0 \Rightarrow (1,6) \notin R$

$\therefore$ Relation is not symmetric.

- (ii) **(d) $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$** [1]

$\sin^{-1}x$ and $\cos^{-1}x$ is defined for $-1 \leq x \leq 1$

$\tan^{-1}x$ is defined for all real x

Given, $x \geq 0$, the valid domain is: $0 \leq x \leq 1$

For all $x \in [-1, 1]$, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$$\theta = \frac{\pi}{2} - \tan^{-1}x$$

The principal value range of $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

But since $x \geq 0$, $\tan^{-1}x \in 0 \left[0, \frac{\pi}{4}\right] (\because x \leq 1)$

$$0 \leq \tan^{-1}x \leq \frac{\pi}{4}$$

$$\Rightarrow -\frac{\pi}{4} \leq -\tan^{-1}x \leq 0$$

$$\Rightarrow \frac{\pi}{4} \leq \frac{\pi}{2} - \tan^{-1}x \leq \frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

(iii) (b) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm 2 A$ [1]

Area of triangle formed by three points is:

$$A = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Since determinant may be positive or negative

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm 2 A$$

(iv) (c) (4,11) and $(-4, -\frac{31}{3})$ [1]

Differentiating w.r.t 't', $6 \frac{dy}{dt} = 3x^2 \frac{dx}{dt}$.

Substituting $\frac{dy}{dt} = 8 \frac{dx}{dt}$ gives the value of $x = \pm 4$.

When $x = 4, y = 11$ and when $x = -4, y = -\frac{31}{3}$

$\therefore$ Points are (4,11) and $(-4, -\frac{31}{3})$

(v) (c) Both the statements are true. [1]

Statement I: If $h'(t) > 0$, it means altitude increases with time, so satellite is moving away from earth

$\therefore$ Statement I is true.

Statement II: At highest point (apogee), altitude becomes maximum. At maximum point, $\frac{dh}{dt} = 0$.

This means change in altitude is zero.

$\therefore$ Statement II is true

(vi) (d) $k = \pm \frac{1}{2\sqrt{2}}$ [1]

$$x = y^2 \Rightarrow 1 = 2y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

$$xy = k \Rightarrow x \frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

Since tangents of the given two curves are at right angles, product of their slopes = -1

$$\Rightarrow \frac{1}{2y} \cdot -\frac{y}{x} = -1 \Rightarrow x = \frac{1}{2} = y^2$$

$$\therefore k^2 = x^2 y^2 \Rightarrow \frac{1}{8}$$

$$\therefore k = \pm \frac{1}{2\sqrt{2}}$$

- (vii) **(c) It is a local minimum because the second derivative is positive for all $x \neq 0$.** [1]

$$\frac{dy}{dx} = 4x^3 \Rightarrow \frac{d^2y}{dx^2} = 12x^2$$

Though $\frac{d^2y}{dx^2} = 0$ when $x = 0$, the fact that $\frac{d^2y}{dx^2} = 12x^2$ is always positive for all $x \neq 0$ which confirms the curve is a concave up at $x = 0$.

The function attains local minimum at $(0,0)$.

- (viii) **(d) $\int_2^8 \left(4 - \frac{x}{2}\right) dx$** [1]

Given equation of line: $x + 2y = 8$

$$\Rightarrow \frac{x}{8} + \frac{y}{4} = 1$$

$$\Rightarrow y = 4 - \frac{x}{2}$$

Required area is the area bounded by $x = 2$; $y = 4 - \frac{x}{2}$ and x -axis.

$$\therefore \text{Required area} = \int_2^8 \left(4 - \frac{x}{2}\right) dx$$

- (ix) **(d) No error** [1]

- (x) **(c) Both of them were correct.** [1]

$$\begin{aligned} \text{Notice that: } \cos\left(-\sin^{-1} x + \frac{\pi}{2}\right) \\ = \sin(\sin^{-1} x) \\ = x \end{aligned}$$

So, the answers are equivalent.

- (xi) **(d) -74** [1]

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ -1 & 2 & -4 \end{vmatrix} = -10\hat{i} + 9\hat{j} + 7\hat{k}$$

$$\vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4\hat{i} - 3\hat{j} - \hat{k}$$

$$\therefore (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c}) = -40 - 27 - 7 = -74$$

- (xii) **(d) $(1, 2, 3)$** [1]

$$\text{Given lines are } \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-1}{5} = \frac{y-2}{2} = \frac{z-3}{1}$$

$\therefore$ Both lines are passing through $(1, 2, 3)$

$\therefore$ The point of intersection is $(1, 2, 3)$.

- (xiii) **(c) 156** [1]

Maximum at points $(8, 10)$ and $(7, 12)$

$$\Rightarrow 8a + 10b = 7a + 12b \Rightarrow a = 2b$$

$$\text{But as } ab = 72 \Rightarrow 2b^2 = 72 \Rightarrow b^2 = 36 \Rightarrow b = 6.$$

$$\therefore a = 2b = 12$$

$$\text{Maximum value} = 8 \times 12 + 10 \times 6 = 156$$

- (xiv) **(b) Statement I is false and Statement II is true.** [1]
 Statement I: Every Linear Programming Problem has at least one optimal Solution.
False - Some LPPs are infeasible
 Statement II: If an Linear Programming Problem has two optimal solutions, then it has infinitely many solutions.
True - If two points give same maximum value, then all points on the line joining them also give same value.
 Hence, infinitely many optimal solutions exist.
- (xv) **(a) $\frac{2m}{1+m}$** [1]

$$P\left(\frac{B}{A}\right) = 6m \Rightarrow P(A \cap B) = P(A) \times 6m = 6m^2$$

$$P\left(\frac{B}{\bar{A}}\right) = 3m \Rightarrow P(B \cap \bar{A}) = P(\bar{A}) \times 3m = 3m(1 - m) = 3m - 3m^2$$

$$P(B) - P(A \cap B) = 3m - 3m^2 \Rightarrow P(B) = 3m - 3m^2 + 6m^2 = 3m + 3m^2$$

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{6m^2}{3m + 3m^2} = \frac{2m}{1+m}$$
- (xvi) **(a) Both Assertion and Reason are true, and Reason is the correct explanation of Assertion.** [1]
 The assertion states that the curves can be considered as $f(x)$ and $f^{-1}(x)$
 Since the reflection property is satisfied
 $\therefore$ Assertion is true.
 The reason states that the curves are reflections of each other across the line $x = y$.
 The reflection of a curve across the line $y = x$ yields the graph of the inverse function because the geometric reflection swaps the x and y coordinates, which corresponds to the algebraic definition of an inverse mapping.
 $\therefore$ Reason is true and Reason correctly explains the Assertion.
- (xvii) **(b) Both Assertion and Reason are true, but Reason is not the correct explanation of Assertion.** [1]
 Let $f(x) = x^3 \cos x$
 $f(-x) = -x^3 \cos x = -f(x)$
 $\Rightarrow f(x)$ is an odd function
 $\Rightarrow \int_{-1}^1 x^3 \cos x \, dx = 0$
 As $\int_{-a}^a f(x) \, dx = 0$ when $f(x)$ is an odd function.
 Hence assertion is true.
 If $f(x)$ is a continuous function defined on $[0, a]$, then
 $\int_0^a f(x) \, dx = \int_0^a f(a - x) \, dx$ is also true but not applicable for this assertion.
- (xviii) **Matrix Multiplication is non – commutative ($AB \neq BA$)** [1]
 $(A + B)(A - B) = A^2 - AB + BA - B^2$
 But $AB \neq BA$
 $\therefore (A + B)(A - B) \neq A^2 - B^2$
 Hence, the concept is incorrect.

(xix) $a = -2$ [1]

$$\frac{dy}{dx} = \frac{ax+3}{2y+5}$$

$$\Rightarrow (2y+5)dy = (ax+3)dx$$

Integrating both sides:

$$\Rightarrow \int (2y+5)dy = \int (ax+3)dx$$

$$\Rightarrow y^2 + 5y + c = a \frac{x^2}{2} + 3x$$

$$\Rightarrow -a \frac{x^2}{2} - 3x + y^2 + 5y + c = 0$$

Now if $a = -2$, then the solution of the differential equation will be

$$\Rightarrow x^2 + y^2 - 3x + 5y + c = 0, \text{ which is an equation of circle.}$$

$$\therefore a = -2$$

(xx) 0.2 [1]

$$P(X \leq 2) = 0.28 \Rightarrow 4m = 0.28 \Rightarrow m = 0.07$$

$$P(X \geq 4) = 0.52 \Rightarrow 5m + b = 0.52 \Rightarrow b = 0.52 - 0.35 = 0.17$$

$$9m + a + b = 1 \Rightarrow a = 1 - 0.63 - 0.17 = 0.2$$

$$P(X = 3) = 0.2$$

SECTION B – 14 MARKS

Question 2 [2]

$$\therefore \vec{r} = 5\hat{i} + \lambda(5\hat{i} + \sin \theta \hat{j} + \cos \theta \hat{k}),$$

$$\therefore \text{Direction vector of above line } \vec{d} = 5\hat{i} + \sin \theta \hat{j} + \cos \theta \hat{k}.$$

Now equation of plane π_p is $x = p$.

$$\Rightarrow \vec{r} \cdot \vec{n} = p \quad \text{where } n \text{ is the normal vector to the plane.}$$

$$\therefore \vec{n} = \hat{i} + 0\hat{j} + 0\hat{k}$$

$$\Rightarrow \vec{n} = \hat{i}$$

$$\therefore \sin \theta = \frac{|\vec{d} \cdot \vec{n}|}{|\vec{d}| \cdot |\vec{n}|} = \frac{|5|}{\sqrt{26} \cdot 1}$$

$$\theta = \sin^{-1} \left(\frac{5}{\sqrt{26}} \right)$$

Hence proved.

Question 3 [2]

On any given day, Tanishka can be either present or absent.

Total possible outcomes = 2^8

If we consider that she comes to school as **P** and she is absent as **A**,

Case 1: If Tanishka comes for **at least** the first 5 consecutive working days, then we just need to consider the attendance for the remaining 3 days.

PPPPP _ _ _

This can be done in 2^3 ways.

Case 2: If Tanishka comes for **at least** any other 5 consecutive working days, then she must be absent on the preceding day. So, we just need to consider the attendance for the remaining 2 days.

APPPPP _ _ or _ **APPPPP** _ or _ _ **APPPPP**

This can be done in 2^2 ways.

∴ Probability (comes to school for at least 5 consecutive working days)

$$= \frac{2^3 + 2^2 + 2^2 + 2^2}{2^8} = \frac{5}{64}$$

Question 4

[2]

$$\frac{dx}{dy} = \frac{x}{y} + \frac{f\left(\frac{x}{y}\right)}{f'\left(\frac{x}{y}\right)}$$

Given differential equation is homogenous differential equation of type

$$\frac{dx}{dy} = f\left(\frac{x}{y}\right)$$

Substituting, $x = vy$ i.e. $\frac{x}{y} = v$

$$\Rightarrow \frac{dx}{dy} = v + y \cdot \frac{dv}{dy}$$

$$\Rightarrow v + y \cdot \frac{dv}{dy} = v + \frac{f(v)}{f'(v)}$$

$$\Rightarrow y \cdot \frac{dv}{dy} = \frac{f(v)}{f'(v)}$$

$$\Rightarrow \frac{f'(v)}{f(v)} dv = \frac{dy}{y}$$

Integrating both sides

$$\Rightarrow \int \frac{d(f(v))}{f(v)} = \int \frac{dy}{y} + \log c$$

$$\Rightarrow \log|f(v)| = \log y + \log c$$

$$\Rightarrow f(v) = cy$$

$$\Rightarrow f\left(\frac{x}{y}\right) = cy, \text{ which is the required solution.}$$

Question 5

- (i) Since differentiability implies continuity, the limits from both sides at $x = 0$ must be equal. [2]

$$LHL = \lim_{x \rightarrow 0^-} p \sin x + q = q$$

$$RHL = \lim_{x \rightarrow 0^+} x^2 + 2x + 1 = 1$$

$$\text{At } x = 0, f(0) = q = LHL = RHL \Rightarrow q = 1$$

The Left-Hand Derivative (LHD) must equal the Right-Hand Derivative (RHD) at $x = 0$.

$$LHD = \frac{d}{dx} (p \sin x + q) = p \cos x,$$

$$\text{At } x = 0, LHD = p \cos x = p$$

$$RHD = \frac{d}{dx} (x^2 + 2x + 1) = 2x + 2$$

At $x = 0$, $RHD = 2x + 2 = 2$
 $\Rightarrow p = 2$

OR

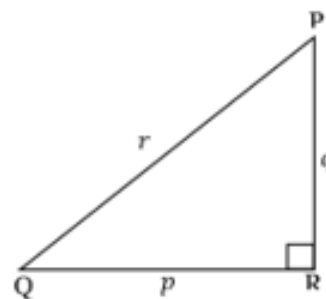
- (ii) (a) At $x = -2$ the function has removable discontinuity. [1]
 At $x = -2$, $LHL = RHL$ but $f(-2)$ is missing.
 (b) $f(x)$ is discontinuous at $x = -2, x = 0$ and $x = 2$ hence not differentiable. [1]

Question 6

[2]

- (i) Given, In ΔPQR , $\angle R = 90^\circ$
 $\therefore r^2 = p^2 + q^2$ (i) (Pythagoras Theorem)
 Now,

$$\begin{aligned} \tan^{-1} \left(\frac{\frac{p}{q+r} + \frac{q}{r+p}}{1 - \frac{pq}{(q+r)(r+p)}} \right) \\ = \tan^{-1} \left(\frac{pr + p^2 + q^2 + qr}{qr + pq + r^2 + pr - pq} \right) \\ = \tan^{-1} \left(\frac{pr + r^2 + qr}{pr + r^2 + qr} \right) \dots \dots \text{(from (i))} \\ = \tan^{-1} (1) \\ = \frac{\pi}{4} \end{aligned}$$



OR

- (ii) Given, $ax^2 + \sin^{-1}(x^2 - 2x + 2) + \cos^{-1}(x^2 - 2x + 2) = 0$
 $\therefore ax^2 + \frac{\pi}{2} = 0$ (i) $\left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \right]$

For the given inverse functions: $-1 \leq x^2 - 2x + 2 \leq 1$

Now, $(x^2 - 2x + 2) = (x - 1)^2 + 1$

Since $(x - 1)^2 \geq 0$

Minimum value occurs when $x = 1$

$\therefore (x - 1)^2 + 1 \geq 1$

To satisfy domain ≤ 1 we have,

$(x - 1)^2 + 1 = 1$

$\therefore (x - 1)^2 = 0$

$\therefore x = 1$

Thus, only one value of x exists

Hence, roots are equal at $x = 1$

$\therefore a(1)^2 + \frac{\pi}{2} = 0$

$\therefore a = -\frac{\pi}{2}$

Question 7**[2]**

- (i) Vector projection of B on A

$$\begin{aligned}
 &= (\vec{B} \cdot \hat{A}) \hat{A} \\
 &= \left(\frac{\vec{B} \cdot \vec{A}}{|\vec{A}|} \right) \hat{A} \\
 &= \left(\frac{\vec{B} \cdot \vec{A}}{|\vec{A}|^2} \right) \vec{A} \\
 &= -\frac{4}{9}(\hat{i} - 2\hat{j} - 2\hat{k})
 \end{aligned}$$

Scalar component of $\vec{B}$ on $\vec{A}$

$$\begin{aligned}
 &= \vec{B} \cdot \frac{\vec{A}}{|\vec{A}|} \\
 &= (6\hat{i} + 3\hat{j} + 2\hat{k}) \cdot \left(\frac{\hat{i}}{3} - \frac{2\hat{j}}{3} - \frac{2\hat{k}}{3} \right) \\
 &= -\frac{4}{3}
 \end{aligned}$$

OR

- (ii)
- $A(\vec{a}) = \hat{i} + \hat{j} + \hat{k}$
- ,
- $B(\vec{b}) = 2\hat{i} + 3\hat{j} - 4\hat{k}$
- ,
- $B(\vec{c}) = 7\hat{i} + 4\hat{j} + 9\hat{k}$

$$\therefore \vec{AB} = \hat{i} + 2\hat{j} - 5\hat{k} \text{ and } \vec{AC} = 6\hat{i} + 3\hat{j} + 8\hat{k}$$

$$\begin{aligned}
 \therefore \vec{AB} \times \vec{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -5 \\ 6 & 3 & 8 \end{vmatrix} \\
 &= 31\hat{i} - 38\hat{j} - 9\hat{k}
 \end{aligned}$$

$$|31\hat{i} - 38\hat{j} - 9\hat{k}| = \sqrt{2486}$$

$$\text{Unit vector} = \frac{31\hat{i} - 38\hat{j} - 9\hat{k}}{\sqrt{2486}}$$

Question 8

- (i) Position vector of IMA(I) relative to Clock Tower (
- $\vec{OI}$
-) =
- $3\hat{i} + 9\hat{j}$

[1]

$$\text{Position vector of Rajpur (R) relative to Clock Tower } (\vec{OR}) = 8\hat{i} + 4\hat{j}$$

$$\begin{aligned}
 \text{Position vector of Bus stop(S) relative to Clock Tower } (\vec{OS}) &= \frac{2}{3}(3\hat{i} + 9\hat{j}) \\
 &= 2\hat{i} + 6\hat{j}
 \end{aligned}$$

- (ii) Now,
- $\vec{RS} = \vec{OS} - \vec{OR} = 2\hat{i} + 6\hat{j} - 8\hat{i} - 4\hat{j} = -6\hat{i} + 2\hat{j}$
- .

[1]

$$\therefore \vec{OS} \cdot \vec{RS} = -12 + 12 = 0.$$

 $\Rightarrow$ Bus stop is the closest point to Rajpur on the Clock Tower to IMA Road.**SECTION C – 21 MARKS****Question 9**

- (i)
- $2x + y \leq 9$

[1]

$$y \geq x$$

$$x \geq 1$$

(ii) $Z = 8$ at $C(1,7)$ [1]

(iii) $2x + y \geq 9$ [1]

Question 10 [3]

$$\frac{dx}{dz} = 3, \text{ and } \frac{dy}{dz} = \frac{dy}{dz} \cdot \frac{dz}{dx} \rightarrow \frac{dy}{dz} = 3 \frac{dy}{dx}$$

$$\frac{d^2y}{dz^2} = 3 \frac{d}{dz} \left(\frac{dy}{dx} \right)$$

$$= 3 \frac{d}{dx} \left(\frac{dy}{dx} \right) \cdot \frac{dx}{dz}$$

$$= 3 \frac{d^2y}{dx^2} \cdot 3$$

$$\frac{d^2y}{dz^2} = 9 \frac{d^2y}{dx^2}$$

Question 11

(i) Simplify the given function: [2]

$$f(x) = \log \left[\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x} \cdot \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}+x} \right] + \tan^{-1} \frac{2x}{1-x^2}$$

$$f(x) = \log \left[\frac{(\sqrt{1+x^2}+x)^2}{1+x^2-x^2} \right] + 2\tan^{-1} x$$

$$f(x) = 2 \log(\sqrt{1+x^2}+x) + 2\tan^{-1} x$$

$$f'(x) = 2 \frac{1}{\sqrt{1+x^2}+x} \cdot \left(\frac{2x}{2\sqrt{1+x^2}} + 1 \right) + \frac{2}{1+x^2}$$

$$f'(x) = \frac{2}{\sqrt{1+x^2}} + \frac{2}{1+x^2}$$

(ii) The claim is incorrect. [1]

Since $x^2 \geq 0$, both $\frac{2}{\sqrt{1+x^2}}$ and $\frac{2}{1+x^2}$ are always positive.

Thus, $f'(x) > 0$ for all real x .

A local maximum requires $f'(x)$ to be zero and change sign. Here, the function is strictly increasing.

Question 12

(i) Let γ be the angle made by $\vec{n}$ with z-axis, then direction cosines of $\vec{n}$ are [3]

$$l = \cos 45^\circ = \frac{1}{\sqrt{2}}, m = \cos 60^\circ = \frac{1}{2} \text{ and } n = \cos \gamma$$

$$\therefore l^2 + m^2 + n^2 = 1$$

$$n^2 = \frac{1}{4}$$

$$n = \frac{1}{2} \text{ (neglecting } n = -\frac{1}{2} \text{ as } \gamma \text{ is acute)}$$

$$\therefore |\vec{n}| = 8$$

$$\therefore \vec{n} = |\vec{n}|(l\hat{i} + m\hat{j} + n\hat{k})$$

$$\vec{n} = 8 \left(\frac{i}{\sqrt{2}} + \frac{j}{2} + \frac{k}{2} \right)$$

$$= 4\sqrt{2}\hat{i} + 4\hat{j} + 4\hat{k}$$

∴ Required plane passes through the point $(\sqrt{2}, -1, 1)$ having position vector

$$\vec{a} = \sqrt{2}\hat{i} - \hat{j} + \hat{k}$$

∴ Vector equation is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

$$\vec{r} \cdot (4\sqrt{2}\hat{i} + 4\hat{j} + 4\hat{k}) = (\sqrt{2}\hat{i} - \hat{j} + \hat{k}) \cdot (4\sqrt{2}\hat{i} + 4\hat{j} + 4\hat{k})$$

$$\Rightarrow \vec{r} \cdot (4\sqrt{2}\hat{i} + 4\hat{j} + 4\hat{k}) = 8$$

$$\Rightarrow \vec{r} \cdot (\sqrt{2}\hat{i} + \hat{j} + \hat{k}) = 2$$

OR

(ii) (a) $\pi_1: 2x + 6y - 2z = 5$ [1]

Substituting the point P $\left(2, \frac{1}{2}, 1\right)$,

$$LHS = 2 \times 2 + 6 \times \frac{1}{2} - 2 \times 1 = 4 + 3 - 2 = 5 = RHS$$

(b) $\pi_2: 3x + 9y + pz = -\frac{51}{2}$ [1]

$$\text{Given that } \pi_2 \parallel \pi_1 \Rightarrow \frac{3}{2} = \frac{9}{6} = \frac{p}{-2}$$

$$\therefore p = -3$$

(c) Equation of the line that is perpendicular to π_1 is [1]

$$\frac{x-2}{1} = \frac{y-\frac{1}{2}}{3} = \frac{z-1}{-1} = \lambda$$

General points on the line are $(\lambda + 2, 3\lambda + \frac{1}{2}, -\lambda + 1)$

∴ Q lies on π_2

$$\Rightarrow 3(\lambda + 2) + 9\left(3\lambda + \frac{1}{2}\right) - 3(1 - \lambda) = -\frac{51}{2}$$

$$\Rightarrow \lambda + 2 + 9\lambda + \frac{3}{2} - 1 + \lambda = -\frac{17}{2}$$

$$\Rightarrow 11\lambda = -11 \Rightarrow \lambda = -1$$

∴ Co-ordinates of Q $\left(1, -\frac{5}{2}, 2\right)$

Question 13

[3]

(i)

$$\Delta = \frac{1}{xyz} \begin{vmatrix} x^2 & x^3 & xyz \\ y^2 & y^3 & xyz \\ z^2 & z^3 & xyz \end{vmatrix} = \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix} \quad \text{Applied the property } |A| = |A^t|$$

$$= - \begin{vmatrix} y^2 & x^2 & z^2 \\ y^3 & x^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} y^2 & z^2 & x^2 \\ y^3 & z^3 & x^3 \\ 1 & 1 & 1 \end{vmatrix} \quad (\text{Hence proved})$$

$$\Delta = \begin{vmatrix} x^2 & x^3 & 1 \\ y^2 & y^3 & 1 \\ z^2 & z^3 & 1 \end{vmatrix}$$

$$R_1 \rightarrow R_1 - R_2 \text{ and } R_2 \rightarrow R_2 - R_3$$

$$\Delta = \begin{vmatrix} x^2 - y^2 & x^3 - y^3 & 0 \\ y^2 - z^2 & y^3 - z^3 & 0 \\ z^2 & z^3 & 1 \end{vmatrix} = (x - y)(y - z) \begin{vmatrix} x + y & x^2 + y^2 + xy & 0 \\ y + z & y^2 + z^2 + yz & 0 \\ z^2 & z^3 & 1 \end{vmatrix}$$

$$\text{Again } R_2 \rightarrow R_2 - R_1$$

$$= (x - y)(y - z) \begin{vmatrix} x + y & x^2 + y^2 + xy & 0 \\ z - x & z^2 - x^2 + y(z - x) & 0 \\ z^2 & z^3 & 1 \end{vmatrix}$$

$$= (x - y)(y - z)(z - x) \begin{vmatrix} x + y & x^2 + y^2 + xy & 0 \\ 1 & x + y + z & 0 \\ z^2 & z^3 & 1 \end{vmatrix}$$

$$= (x - y)(y - z)(z - x)(xy + yz + zx)$$

$$= (x - y)(y - z)(z - x) \quad (\text{As } xy + yz + zx = 1)$$

OR

(ii) Let Q be the quantities of chair ribbon and table cover needed

$$\therefore Q = \begin{bmatrix} 150 & 20 \end{bmatrix}$$

Let P represents the prices of chair ribbon and table cover from each company.

$$\therefore P = \begin{bmatrix} 12 & 10 \\ 180 & 200 \end{bmatrix} \begin{matrix} \leftarrow \text{chair ribbons} \\ \leftarrow \text{table cover} \end{matrix}$$

$\begin{matrix} \uparrow & \uparrow \\ \text{Company A} & \text{Company B} \end{matrix}$

$\therefore$ Total cost matrix is given by

$$C = QP$$

$$\therefore C = \begin{bmatrix} 150 & 20 \end{bmatrix} \begin{bmatrix} 12 & 10 \\ 180 & 200 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} 150 \times 12 + 20 \times 180 & 150 \times 10 + 20 \times 200 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} 1800 + 3600 & 1500 + 4000 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} 5400 & 5500 \end{bmatrix}$$

Thus, the price of company A (₹ 5400) is ₹ 100 less than the price of the company B (₹5500).

Question 14

(i) Given, $\int_1^5 3f(x)dx = 12$

[1]

$$\Rightarrow \int_1^5 f(x)dx = 4$$

$$\Rightarrow -\int_1^5 f(x)dx = -4$$

$$\Rightarrow \int_5^1 f(x)dx = -4. \quad \left[\because \int_a^b f(x)dx = -\int_b^a f(x)dx \right]$$

Hence proved.

(ii)
$$\int_1^2 (x + f(x)) dx + \int_2^5 (x + f(x)) dx$$

$$= \int_1^5 (x + f(x)) dx$$

$$= \int_1^5 x dx + \int_1^5 f(x) dx$$

$$= \left[\frac{x^2}{2} \right]_1^5 + 4$$

$$= \frac{25}{2} - \frac{1}{2} + 4$$

$$= 16$$

[2]

Question 15

- (i) Given, L is the tangent to the graph of f at the origin O.

Slope of the tangent at O = $f'(0)$

$$= \frac{2a^2 - 0}{\sqrt{a^2 - 0}} = 2a$$

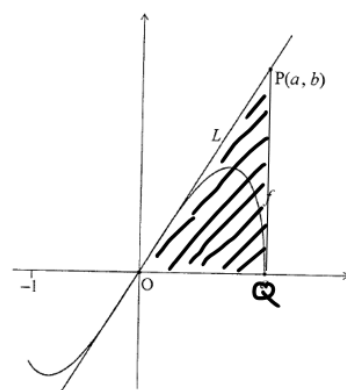
$\therefore$ Equation of L: $y - 0 = 2a(x - 0)$

$$\Rightarrow y = 2ax.$$

$$\therefore \Delta OPQ = \int_0^a 2ax dx.$$

$$= 2a \times \left[\frac{x^2}{2} \right]_0^a$$

$$= a^3 \text{ sq. units.}$$



[3]

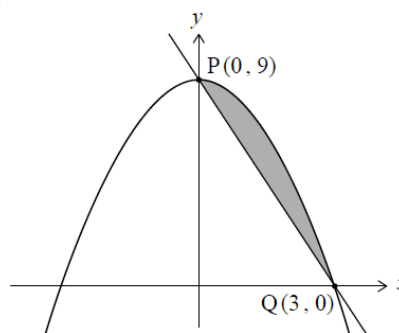
- (ii) (a) Putting $x=0$ in $y = -3x + 9$, $y=9$
 $\Rightarrow P(0,9)$
 and Putting $y=0$ in $y = -3x + 9$, $x=3$
 $\Rightarrow Q(3,0)$

The parabola is of the form $y = ax^2 + c$, where $a, c \in \mathbb{Z}$

Satisfying by, $(0,9)$, we get $c = 9$.

Satisfying by, $(3,0)$, we get $a = -1$.

$\therefore$ Equation of parabola is: $y = -x^2 + 9$



[1]

- (b) Equation of line PQ is: $\frac{x}{3} + \frac{y}{9} = 1$ i.e. $y = 9 - 3x$

$\therefore$ The area of the shaded region

$$= \int_0^3 (9 - x^2 - 9 + 3x) dx$$

$$= \int_0^3 (3x - x^2) dx$$

[2]

$$\begin{aligned}
 &= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 \\
 &= \frac{27}{2} - 9 \\
 &= \frac{9}{2} \text{ sq units}
 \end{aligned}$$

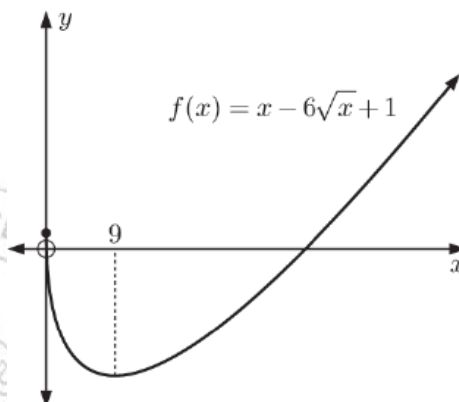
SECTION D – 25 MARKS

Question 16

(i) $f(x) = x - 6\sqrt{x} + 1$

Natural domain of $f(x)$ is $x \geq 0$.

i.e. $[0, \infty)$



[1]

(ii) As the graph fails the horizontal line test, hence it is not one-one.

[1]

$\Rightarrow f(x)$ is not invertible.

$$f(x) = x - 6\sqrt{x} + 1$$

$$\Rightarrow y = x - 6\sqrt{x} + 1$$

$$\Rightarrow (\sqrt{x})^2 - 6\sqrt{x} + (1 - y) = 0$$

$$\therefore \sqrt{x} = \frac{6 \pm \sqrt{36 - 4(1-y)}}{2} = \frac{6 \pm 2\sqrt{y+8}}{2} = 3 \pm \sqrt{y+8}$$

$$\therefore x = (3 \pm \sqrt{y+8})^2$$

(iii) Given, $g(x) = x - 6\sqrt{x} + 1, 0 \leq x \leq 9$.

[1]

$$\therefore g^{-1}(x) = (3 - \sqrt{x+8})^2$$

$$\therefore (g \circ g^{-1})(x) = (3 - \sqrt{x+8})^2 - 6(3 - \sqrt{x+8}) + 1$$

$$= 9 - 6\sqrt{x+8} + x + 8 - 18 + 6\sqrt{x+8} + 1 = x$$

$$\therefore (g \circ g^{-1})(4) = 4$$

(iv) Given, $h(x) = x - 6\sqrt{x} + 1, x \geq 9$.

[1]

$$\therefore h^{-1}(x) = (3 + \sqrt{x+8})^2$$

$$\therefore (h \circ h^{-1})(x) = (3 + \sqrt{x+8})^2 - 6(3 + \sqrt{x+8}) + 1$$

$$= 9 + 6\sqrt{x+8} + x + 8 - 18 - 6\sqrt{x+8} + 1 = x$$

$$\therefore (h \circ h^{-1})(16) = 16$$

(v) When $g^{-1}(x) = h^{-1}(x)$ [1]

$$\Rightarrow (3 - \sqrt{x+8})^2 = (3 + \sqrt{x+8})^2$$

$$\Rightarrow 9 - 6\sqrt{x+8} + x + 8 = 9 + 6\sqrt{x+8} + x + 8$$

$$\Rightarrow \sqrt{x+8} = 0$$

$$\Rightarrow x = -8$$

Question 17

- (i) (a) $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 5$ [1]
- $$f'(x) = x^2 - 4x + 3$$
- $$f'(x) = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1 \text{ \& } x = 3$$
- Climbing up (increasing) $\Rightarrow f'(x) > 0$ on $[0,1) \cup (3, \infty)$
- Descending (decreasing) $\Rightarrow f'(x) < 0$ on $(1,3)$
- (b) Slope of the horizontal tangent $f'(x) = 0$, at $x = 1$ & $x = 3$ [1]
- At $x = 1$, $f(1) = \frac{1}{3} - 2 + 3 + 5 = 6.33$
- At $x = 3$, $f(3) = 9 - 18 + 9 + 5 = 5$
- Coordinates of the points on the curve: $(1, 6.33)$ & $(3, 5)$
- (c) Maxima & Minima in the interval $[0, 4]$ [2]
- Evaluate the function at the critical points $x = 1$ and $x = 3$ and end points of the interval $x = 0$ and $x = 4$
- $$f(0) = 5$$
- $$f(1) = 6.33$$
- $$f(3) = 5$$
- $$f(4) = \frac{64}{3} - 32 + 12 + 5 = 6.33$$
- The absolute maximum is 6.33 at $x = 1$ and $x = 4$
- The absolute minimum is 5 at $x = 0$ and $x = 3$
- (d) The equation of normal at $x = 1$ [1]
- Slope of the tangent at $x = 1 \Rightarrow f'(1) = 0$
- i.e. the tangent is a horizontal line $\Rightarrow$ normal is a vertical line ($x = 1$)

OR

- (ii) Let the radius, height, volume and semi vertical angle of the cone be r, h, V & α
- $$\tan \alpha = \frac{r}{h} = 0.5 \Rightarrow r = 0.5h$$
- Rate of flow (drained out) $= \frac{dV}{dt} = -5m^3/min$
- Volume $= V = \frac{1}{3}\pi r^2 h$
- (a) $V = \frac{1}{3}\pi(0.5h)^2 h = \frac{0.25}{3}\pi h^3 = \frac{\pi}{12}h^3$ [2]
- $$\frac{dV}{dt} = \frac{\pi}{12} 3h^2 \frac{dh}{dt} \Rightarrow \frac{\pi}{4} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{\pi}{4}h^2} = \frac{-5}{\frac{\pi}{4}h^2}$$
- Now substitute $\frac{dV}{dt} = -5$ and $h = 4$
- $$-5 = \frac{\pi}{4} 4^2 \frac{dh}{dt} = 4\pi \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{-5}{4\pi} m/min$$

$$(b) \quad l = \text{Slant height} = \sqrt{r^2 + h^2} = \sqrt{(0.5h)^2 + h^2} = \sqrt{1.25h^2} = \frac{\sqrt{5}}{2}h \quad [1]$$

$$\text{Curved Surface area} = \pi r l$$

$$S = \pi(0.5h) \frac{\sqrt{5}}{2}h = \frac{\sqrt{5}}{4}\pi h^2$$

$$\frac{dS}{dh} = \frac{\sqrt{5}}{2}\pi h$$

Since $h > 0$ for physical tank $\frac{dS}{dh} > 0$

$\Rightarrow$ Surface area S is strictly increasing on $(0, \infty)$

$$(c) \quad \text{Given that Cost of the additive } C = kS^2 = k\left(\frac{\sqrt{5}}{4}\pi h^2\right)^2 = \frac{5k\pi^2}{16}h^4 \quad [2]$$

$$\begin{aligned} \text{Rate of change of cost} &= \frac{dC}{dt} = \frac{dC}{dh} \cdot \frac{dh}{dt} \\ &= \frac{5k\pi^2}{4}h^3 \cdot \frac{5}{4}h^2 = 25k\pi h \end{aligned}$$

The rate of cost change $\frac{dC}{dt}$ is a linear function of h .

The rate of change increases as h increases. Therefore, the cost is increasing most rapidly at the maximum possible depth h allowed by the tank.

Question 18

$$(i) \quad (a) \quad \int \ln x \, dx \quad [2]$$

$$= \int \ln x \cdot 1 \, dx$$

I II

Applying by parts:

$$= \ln x \cdot \int 1 \, dx - \int \left\{ \frac{d}{dx}(\ln x) \int 1 \, dx \right\} dx$$

$$= \ln x \times x - \int \frac{1}{x} \times x \, dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x(\ln x - 1) + c$$

$$(b) \quad I = \int \frac{\ln\left[\ln\left(\frac{1+x}{1-x}\right)\right]}{1-x^2} dx \quad [3]$$

$$\text{Let } \ln\left(\frac{1+x}{1-x}\right) = t$$

Differentiating w.r.t 't':

$$\Rightarrow \frac{1-x}{1+x} \times \frac{1-x-(1+x)(-1)}{(1-x)^2} \times \frac{dx}{dt} = 1$$

$$\Rightarrow 2 \times \frac{1}{1-x^2} dx = dt$$

$$\therefore I = \int \frac{\ln\left[\ln\left(\frac{1+x}{1-x}\right)\right]}{1-x^2} dx$$

$$= \frac{1}{2} \int \ln\left[\ln\left(\frac{1+x}{1-x}\right)\right] \times \frac{2}{1-x^2} dx$$

$$= \frac{1}{2} \int \ln t \cdot dt$$

$$= \frac{1}{2} t(\ln t - 1) + c$$

$$= \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \left[\ln\left[\ln\left(\frac{1+x}{1-x}\right)\right] - 1 \right] + c$$

OR

$$\begin{aligned}
 \text{(ii) (a)} \quad & \int \tan x \, dx \\
 &= \int \frac{\sin x}{\cos x} \, dx \\
 &= - \int \frac{d(\cos x)}{\cos x} \\
 &= -\ln|\cos x| + c
 \end{aligned}
 \tag{2}$$

$$\begin{aligned}
 \text{(b)} \quad & I = \int \frac{dx}{\cot^{\frac{x}{2}} \cot^{\frac{x}{3}} \cot^{\frac{x}{6}}} \\
 &= \int \tan^{\frac{x}{2}} \tan^{\frac{x}{3}} \tan^{\frac{x}{6}} dx \\
 \text{Now, } & \frac{x}{2} = \frac{x}{3} + \frac{x}{6} \\
 \Rightarrow & \tan^{\frac{x}{2}} = \tan\left(\frac{x}{3} + \frac{x}{6}\right) \\
 \Rightarrow & \tan^{\frac{x}{2}} = \frac{\tan^{\frac{x}{3}} + \tan^{\frac{x}{6}}}{1 - \tan^{\frac{x}{3}} \tan^{\frac{x}{6}}} \\
 \Rightarrow & \tan^{\frac{x}{2}} \tan^{\frac{x}{3}} \tan^{\frac{x}{6}} = \tan^{\frac{x}{2}} - \tan^{\frac{x}{3}} - \tan^{\frac{x}{6}} \\
 \therefore & I = \int \tan^{\frac{x}{2}} dx - \int \tan^{\frac{x}{3}} dx - \int \tan^{\frac{x}{6}} dx \\
 &= -2 \ln \left| \cos \frac{x}{2} \right| + 3 \ln \left| \cos \frac{x}{3} \right| + 6 \ln \left| \cos \frac{x}{6} \right| + c
 \end{aligned}
 \tag{3}$$

Question 19

$$\text{(i)} \quad P(\text{laptop behind door A}) = P(\text{laptop behind door B}) = P(\text{laptop behind door C}) = \frac{1}{3}$$

If the laptop is behind door A, either door B or door C can be chosen to be opened.

$$\Rightarrow P(\text{Door B opened/ laptop behind A}) = \frac{1}{2}$$

Since the host will never reveal the laptop

$$\Rightarrow P(\text{Door B opened/ laptop behind B}) = 0$$

If the laptop is behind door C, then only door B can be opened since door A is the selection of the contestant and cannot be opened.

$$\Rightarrow P(\text{Door B opened/ laptop behind C}) = 1$$

$$\text{(a)} \quad P(\text{laptop is behind Door A/Door B has been opened}) = \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{1}{3} \tag{2}$$

$$\text{(b)} \quad P(\text{laptop is behind Door C/Door B has been opened}) = \frac{1 \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{2}{3} \tag{2}$$

$$\text{(c)} \quad \text{The contestant should change their selection to Door C as the probability is more.} \tag{1}$$

OR

(ii) A: Email is harmful

B: The filter marks email dangerous

$$\text{Probability that the email is harmful} = P(A) = \frac{1}{1500}$$

$$\text{(a)} \quad \text{Probability that the email is safe} = P(\bar{A}) = \frac{1499}{1500} \tag{1}$$

- (b) Probability that given the email is harmful the filter marks it dangerous [2]
 $= P\left(\frac{B}{A}\right) = 0.97$

Probability that the given email is safe and the filter marks it safe $= P\left(\frac{\bar{B}}{\bar{A}}\right) = 0.97$

Probability that the given email is harmful and the filter marks it safe
 $= P\left(\frac{B}{\bar{A}}\right) = 1 - P\left(\frac{B}{A}\right) = 0.03$

- (c) Probability that filter marks the email dangerous (Total probability) [2]
 $= P(B) = P\left(\frac{B}{A}\right)P(A) + P\left(\frac{B}{\bar{A}}\right)P(\bar{A}) = 0.97 \times \frac{1}{1500} + 0.03 \times \frac{1499}{1500}$
 $= 0.0306$

Probability that given the email is marked dangerous but it is safe =

$$P\left(\frac{\bar{A}}{B}\right) = \frac{P(\bar{A} \cap B)}{P(B)} = \frac{P\left(\frac{B}{\bar{A}}\right)P(\bar{A})}{P(B)} = \frac{P\left(\frac{B}{\bar{A}}\right)P(\bar{A})}{P\left(\frac{B}{A}\right)P(A) + P\left(\frac{B}{\bar{A}}\right)P(\bar{A})}$$

$$= \frac{0.03 \times \frac{1499}{1500}}{0.97 \times \frac{1}{1500} + 0.03 \times \frac{1499}{1500}} = 0.9789 \approx 0.98$$

Question 20

[5]

Given, $S(t) = at^2 + bt + c$

$$\therefore S(1) = a(1)^2 + b(1) + c \Rightarrow a + b + c = 285 \rightarrow (i)$$

$$S(2) = a(2)^2 + b(2) + c \Rightarrow 4a + 2b + c = 260 \rightarrow (ii)$$

$$S(4) = a(4)^2 + b(4) + c \Rightarrow 16a + 4b + c = 180 \rightarrow (iii)$$

The system can be written as $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 285 \\ 260 \\ 180 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{bmatrix}$, $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $B = \begin{bmatrix} 285 \\ 260 \\ 180 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix} = 1(2 - 4) - 1(4 - 16) + 1(16 - 32)$$

$$= -2 + 12 - 16 = -6 \neq 0$$

$$\therefore A^{-1} \text{ exists and } A^{-1} = \frac{1}{|A|} \cdot \text{adj.}A.$$

$$\text{adj.}A = \begin{bmatrix} -2 & 3 & -1 \\ 12 & -15 & 3 \\ -16 & 12 & -2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj.}A = -\frac{1}{6} \begin{bmatrix} -2 & 3 & -1 \\ 12 & -15 & 3 \\ -16 & 12 & -2 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -\frac{1}{6} \begin{bmatrix} -2 & 3 & -1 \\ 12 & -15 & 3 \\ -16 & 12 & -2 \end{bmatrix} \begin{bmatrix} 285 \\ 260 \\ 180 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -\frac{1}{6} \begin{bmatrix} -570 + 780 - 180 \\ 3420 - 3900 + 540 \\ -4560 + 3120 - 360 \end{bmatrix} = -\frac{1}{6} \begin{bmatrix} 30 \\ 60 \\ -1800 \end{bmatrix}$$

$\therefore a = -5, b = -10$ and $c = 300$

$\therefore$ Height equation: $S(t) = -5t^2 - 10t + 300$

Height of the cliff:

At $t = 0$, $S(0) = -5(0)^2 - 10(0) + 300$

$\therefore$ Height of the cliff = 300 m.

